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The Transmission of Electric Waves over the Surface of the Earth

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IV. *The Transmission of Electric Waves over the Surface of the Earth.*

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CONTENTS.

1. EVER since the time, about 1902, when MARCONI first succeeded in sending wireless signals across the Atlantic the question of explaining the mechanism of such transmission has attracted attention among mathematicians. The question may be put in the following form :—The electric waves generated by the sending apparatus differ from waves of light only by having a longer wave-length, which is, nevertheless small compared with the radius of the earth ; and the curved surface of the earth may therefore be expected to form a sort of shadow, effectively screening the receiving apparatus at a distance. How, then, does it happen that in practice the waves penetrate into the region of the shadow ? Unfortunately, the question has been investigated by different methods without adequate co-ordination, and the results

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that have been obtained are somewhat discordant. In these circumstances it appears to be desirable to undertake a critical survey of the question.

The various theoretical investigations may be classified as developments of three suggestions: (1) The imperfectly conducting quality, or resistance, of the material, generally sea-water, over which the transmission takes place, may cause the effect observable at a distance to be greater than it would be if the material were perfectly conducting. (2) Owing to the numerical relations connecting the actual wave-lengths used in practice, the size of the earth, and the distances involved, the amount of diffraction, even in the case of perfect conduction, may be greater than would, at first sight, be expected. (3) Transmission through the atmosphere may be notably different from transmission through a homogeneous dielectric. We may refer to these suggestions briefly as the "resistance theory," the "diffraction theory," and the "atmospheric theory." It may be said at once that the atmospheric theory has arisen from the alleged failure of the other two, and that it has not yet been formulated in such a way as to admit of being tested in the same precise analytical fashion as they can. It is still rather speculative and indefinite. In what follows I propose to attend chiefly to the first two suggestions, and to investigate the result that can be obtained by combining them.

To facilitate reference the following list of some of the principal writings on the subject is prefixed. These will hereafter be cited by the numbers placed before them, thus: "MACDONALD (1)." The list does not pretend to be complete:—

(1) MACDONALD, H. M., 'Proc. Roy. Soc.,' vol. 71 (1903), p. 251.
(2) RAYLEIGH, Lord, 'Proc. Roy. Soc.,' vol. 72 (1904), p. 40.
(3) POINCARÉ, H., 'Proc. Roy. Soc.,' vol. 72 (1904), p. 42.
(4) MACDONALD, H. M., 'Proc. Roy. Soc.,' vol. 72 (1904), p. 59.
(5) ZENNECK, J., 'Ann. d. Phys.' (4te Folge), Bd. 23 (1907), p. 846.
(6) SOMMERFELD, A., 'Ann. d. Phys.' (4te Folge), Bd. 28 (1909), p. 665.
(7) MACDONALD, H. M., 'Roy. Soc. Phil. Trans.' (Ser. A.), vol. 210 (1911), p. 113.
(8) POINCARÉ, H., 'Rend. Circ. Mat. Palermo,' t. 29 (1910), p. 169.
(9) NICHOLSON, J. W., 'Phil. Mag.' (Ser. 6), vol. 19 (1910), p. 516; vol. 20 (1910), p. 157; vol. 21 (1911), pp. 62, 281.
(10) NICHOLSON, J. W., 'Phil. Mag.' (Ser. 6), vol. 19 (1910), p. 757.
(11) MACDONALD, H. M., 'Proc. Roy. Soc.' (Ser. A), vol. 90 (1914), p. 50.
(12) MARCH, H. W., 'Ann. d. Phys.' (4te Folge), Bd. 37 (1912), p. 29.
(13) RYBCZYŃSKI, W. VON, 'Ann. d. Phys.' (4te Folge), Bd. 41 (1913), p. 191.
(14) AUSTIN, L. W., 'Bulletin of the Bureau of Standards (Washington),' vol. 7, No. 3 (1911), p. 315.
(15) HOGAN, J. L., 'Electrician,' August 8, 1913.
(16) ECCLES, W. H., 'Proc. Roy. Soc.' (Ser. A), vol. 87 (1912), p. 79.
(17) "Report of a Discussion," 'Brit. Assoc. Rep.,' 1912, p. 401.
(18) ZENNECK, J., 'Lehrbuch der Drahtlosen Telegraphie,' 2te Aufl. (Stuttgart, 1913).

2. In order to simplify the problem and render it definite, certain assumptions are usually made. These may be stated as follows: (1) The earth is taken to be a

homogeneous conductor, surrounded by homogeneous dielectric, the separating surface being a perfect sphere. (2) The sending apparatus is represented by an ideal Hertzian oscillator, or vibrating electric doublet, situated in the dielectric near to the separating surface, and having its axis directed normally to that surface. (3) The waves emitted by the oscillator are taken to be an infinite train of simple harmonic oscillations of a definite frequency. The problem is to determine, in accordance with these assumptions, the electric and magnetic forces at points in the dielectric, which are near to the separating surface but not near to the oscillator. If this problem were solved satisfactorily we should be in a better position for estimating the degree of success attained by the resistance theory and the diffraction theory; but it is precisely in regard to this problem that discordant results have been obtained. This unfortunate state of things has arisen partly from the attempt to separate the effects of resistance from those of curvature. With a view to ascertaining the effect of resistance it has been proposed to simplify the problem still further by treating the surface of the earth, in the first instance, as plane, and afterwards attempting to estimate the modification of the results that would be necessary in order to take account of the curvature. When the effect of curvature is being investigated it is usual to regard the material of the earth, in the first instance, as perfectly conducting, and afterwards to estimate the correction due to resistance. Thus we have a division of the problem into two: the problem of the imperfect conductor with a plane surface, and the problem of the perfect conductor with a spherical surface. It will appear in the sequel that this division of the problem is unnecessary.

3. Current ideas on the subject have been much influenced by the results of two simple limiting cases of the general problem. In one of these the material is considered as perfectly conducting, the separating surface as plane, and the originating doublet as situated on the surface. In this case the waves in the dielectric are exactly the same as if there were no conductor.* The amplitude is subject to diminution through spherical divergence only, so that at a distance from the originating doublet it is inversely proportional to the distance.

In the other limiting case† the distance from the originating doublet is supposed to be so great that the waves can be treated as plane, and the separating surface is also taken to be plane. Then, owing to resistance, the planes of the waves are slightly inclined to the plane boundary, energy being continually supplied from the dielectric to maintain the alternating currents in the conductor, and thus the amplitude of the waves in the dielectric is subject to diminution expressed by a factor of the form e^{-Ax}, where x is a co-ordinate measured along the plane boundary in the direction of propagation. The constant A depends on the resistance and specific inductive capacity of the conductor, and on the wave-length. For low resistances, such as that of sea-water, and large wave-lengths, such as one or more kilometres, it is nearly

* *Cf.* J. A. FLEMING, 'The Principles of Electric Wave Telegraphy,' London, 1906, p. 347.

† See ZENNECK([5]).

proportional to the resistance and to the inverse square of the wave-length. For higher resistances and shorter waves the specific inductive capacity of the conductor affects the value of A sensibly. For sea-water under air, and a wave-length of 5 km., the value of 1/A, the distance in which the amplitude of the waves is diminished in the ratio $1 : e$, is $4{\cdot}78 \times 10^5$, lengths being measured in kilometres. From the discussion of this limiting case of the general problem it has been inferred that increased resistance would be unfavourable to long-distance transmission, while increased wave-length would be very favourable.

4. Returning now to the general problem stated in § 2, we shall take the axis of the doublet to be the axis of z. The system being symmetrical about this axis, it is appropriate to use the function Π introduced by HERTZ,* and sometimes called the Hertzian function.

Let ρ, z, ϕ be cylindrical co-ordinates, the senses of increase of z and ϕ being those of translation and rotation in a right-handed screw, and ρ denoting distance from the axis of z. Let E_ρ, E_z, E_ϕ denote the components of electric force, measured in electrostatic units, and estimated in the directions of increase of ρ, z, ϕ; and let H_ρ, H_z, H_ϕ denote the components of magnetic force, measured in electromagnetic units, and estimated in the same directions. From the symmetry we have the equations

$$E_\phi = H_\rho = H_z = 0. \qquad (1)$$

It will be sufficient to consider the case in which both media are of magnetic capacity unity, the dielectric is of specific inductive capacity unity, and the specific inductive capacity of the conductor is neglected. Then one of the electromagnetic equations is, in both media,

$$-\frac{1}{C}\frac{\partial H_\phi}{\partial t} = \frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho}, \qquad (2)$$

where C is the velocity of light, 3×10^{10} cm. per second. The remaining equations are,

in the dielectric,

$$\frac{1}{C}\frac{\partial E_\rho}{\partial t} = -\frac{\partial H_\phi}{\partial z}, \quad \frac{1}{C}\frac{\partial E_z}{\partial t} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho H_\phi); \qquad (3)$$

in the conductor,

$$4\pi\sigma C E_\rho = -\frac{\partial H_\phi}{\partial z}, \quad 4\pi\sigma C E_z = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho H_\phi); \qquad (4)$$

where σ is the specific conductivity, measured in electromagnetic units.

Now we are to suppose that E_ρ, E_z, H_ϕ, in so far as they depend upon t, are proportional to simple harmonic functions of period $2\pi/kC$ say, where $2\pi/k$ is the wave-length, and we may take them to be proportional to e^{ikCt}. Then they can be

* H. HERTZ, 'Electric Waves,' English ed., p. 140.

expressed in terms of a single function Π, the Hertzian function. The formulæ which hold in the dielectric are

$$H_\phi = -ik\frac{\partial \Pi}{\partial \rho}, \quad E_\rho = \frac{\partial^2 \Pi}{\partial \rho\, \partial z}, \quad E_z = -\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \Pi}{\partial \rho}\right) \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where Π satisfies the equation

$$(\nabla^2 + k^2)\,\Pi = 0\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and those which hold in the conductor are

$$H_\phi = -4\pi\sigma C\frac{\partial \Pi'}{\partial \rho}, \quad E_\rho = \frac{\partial^2 \Pi'}{\partial \rho\, \partial z}, \quad E_z = -\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \Pi'}{\partial \rho}\right), \quad . \quad . \quad . \quad . \quad (7)$$

where Π' satisfies the equation

$$(\nabla^2 + k'^2)\,\Pi' = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

in which

$$k'^2 = -i4\pi\sigma k C, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and Π' has been written instead of Π in the formulæ relating to the conductor.

The special form of Π which answers to a vibrating electric doublet, situated in the dielectric on the axis of z, and having its axis directed along the axis of z, is Π_0 say, where

$$\Pi_0 = \frac{e^{ik(Ct-R)}}{R}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

R denoting distance from the doublet. We may put

$$\Pi = \Pi_0 + \Pi_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Then Π_1 satisfies the same equation (6) as Π.

The conditions to be satisfied by the functions Π_1 and Π' are the following :—

(i.) They are solutions of the equations $(\nabla^2 + k^2)\,\Pi_1 = 0$ and $(\nabla^2 + k'^2)\,\Pi' = 0$;

(ii.) Π_1 is free from singularities in the region outside the conductor, and Π' is free from singularities in the region inside the conductor;

(iii.) Π_1 must represent waves travelling outwards;

(iv.) The tangential components of electric and magnetic force derived, as above, from Π and Π', must be continuous at the bounding surface of the conductor.

5. When the boundary is a plane, say $z = 0$, the conditions (iv.) become

$$\left.\begin{aligned} k^2\Pi &= k'^2\Pi' \\ \frac{\partial \Pi}{\partial z} &= \frac{\partial \Pi'}{\partial z}, \end{aligned}\right\}$$

these equations being satisfied at $z = 0$. The condition (iii.) requires some modification, for both regions now extend to infinite distances. It is now necessary that Π_1

should tend to zero for large positive values of z, the positive sense of the axis of z being directed from the conductor to the dielectric, and that Π' should tend to zero for large negative values of z; and further, that at great distances from the axis of z, Π_1 and Π' should tend to forms which represent waves travelling outwards from that axis.

The problem in a slightly different form was solved by SOMMERFELD.(6) He took the doublet to be situated on the plane boundary, so that Π', as well as Π, has a singularity on this surface; and he obtained an exact solution in terms of definite integrals involving BESSEL'S functions, and devised methods of evaluating the integrals approximately. His main result is an approximate expression for Π_1 at a point close to the boundary, and at a distance ρ from the doublet, in the form

$$\Pi_1 = \Pi_0 i \sqrt{(\tfrac{1}{2}\pi)}\,(k/|k'|)\sqrt{(k\rho)}. \qquad (12)$$

This expression gives a valid approximation if $k\rho$ is not small and not too great; but, as ρ increases, the value of Π_1/Π_0 does not increase so rapidly as this formula indicates, and for very great values of ρ it tends to zero. Within the region of validity of the formula the absolute value of Π_1 diminishes according to the inverse square root of the distance; and the effect represented by Π_1, being a wave affected by cylindrical divergence, is described as a "surface wave." This surface wave is the effect of resistance; and it appears that, owing to resistance, the electric and magnetic forces diminish less rapidly with increasing distance than they would if there were perfect conduction. SOMMERFELD (6) and (17) maintained that this effect of resistance would probably be intensified by curvature of the surface, and might thus counteract the tendency of the signals to become enfeebled owing to curvature. With a wave-length of 5 km. and the conductivity of sea-water ($\sigma = 10^{-11}$) the region of validity of the formula would include distances of 1000 to 10,000 km., and in this region the ratio $|\Pi|/|\Pi_0|$ would increase regularly from 1·003 to 1·032.

The value of Π_1/Π_0 depends upon the resistance and the wave-length as well as upon the distance ρ. It varies directly as the square root of the resistance and inversely as the wave-length. The formula (12) indicates that in the region within which it gives a good approximation, increased resistance is favourable to long-distance transmission, increased wave-length unfavourable. These results are directly opposed to those which were noted in § 3, as derived from the study of the limiting case in which the waves are treated as plane. On the other hand, the range of values of ρ within which the formula is valid becomes narrower as the resistance increases or the wave-length diminishes. The optical theory of shadows shows that increased wave-length must be favourable to long-distance transmission, and we see that we cannot hope to obtain any equivalent result from the solution of the plane problem. It is, however, quite feasible that resistance also should be favourable within a restricted range. The only way to settle the question is to solve the problem of the spherical conductor, supposed imperfectly conducting.

6. It appears to be desirable to write out the analysis for the problem of the spherical conductor rather fully, in order to show how to determine the effect of resistance, and to criticise the attempts that have been made to determine the effect of curvature in the absence of resistance.

We denote the radius of the sphere by a, and use polar co-ordinates r, θ, ϕ, with the centre of the sphere as origin, and the radius vector on which the originating doublet lies as the axis $\theta = 0$. We write μ for $\cos\theta$, and note the formulæ

$$\rho = r\sin\theta, \qquad \rho\frac{\partial}{\partial\rho} = (1-\mu^2)\left(r\frac{\partial}{\partial r} - \mu\frac{\partial}{\partial\mu}\right). \quad . \quad . \quad . \quad . \quad . \quad (13)$$

We denote the components of electric force in the directions of increase of r, θ, ϕ by E_r, E_θ, E_ϕ, and those of magnetic force by H_r, H_θ, H_ϕ. In both media we have

$$E_\phi = H_r = H_\theta = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

In the dielectric, where $r > a$, we may put

$$H_\phi = -ik\frac{\partial\Pi}{\partial\rho}, \qquad E_\theta = \frac{1}{\rho}\frac{\partial}{\partial r}\left(\rho\frac{\partial\Pi}{\partial\rho}\right), \qquad E_r = \frac{1}{r^2}\frac{\partial}{\partial\mu}\left(\rho\frac{\partial\Pi}{\partial\rho}\right); \quad . \quad . \quad (15)$$

and in the conductor, where $r < a$, we may put

$$H_\phi = -4\pi\sigma C\frac{\partial\Pi'}{\partial\rho}, \qquad E_\theta = \frac{1}{\rho}\frac{\partial}{\partial r}\left(\rho\frac{\partial\Pi'}{\partial\rho}\right), \qquad E_r = \frac{1}{r^2}\frac{\partial}{\partial\mu}\left(\rho\frac{\partial\Pi'}{\partial\rho}\right) \quad . \quad . \quad (16)$$

Then Π and Π' satisfy the differential equations (6) and (8).

The conditions which hold at the boundary $r = a$ are

$$\left.\begin{aligned} k^2\frac{\partial\Pi}{\partial\rho} &= k'^2\frac{\partial\Pi'}{\partial\rho}, \\ \frac{\partial}{\partial r}\left(\rho\frac{\partial\Pi}{\partial\rho}\right) &= \frac{\partial}{\partial r}\left(\rho\frac{\partial\Pi'}{\partial\rho}\right). \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The function Π_0 answering to the primary waves is given by (10), and we take the originating doublet, the origin of R, to be at the point for which $r = r_0$, $\theta = 0$. Then $r_0 > a$, but in practically interesting cases $(r_0 - a)/a$ is small. The functions Π_1 and Π' are to be determined in accordance with the conditions laid down in § 4.

The proper forms for these functions can be expressed as series involving spherical harmonics, viz.:

$$\left.\begin{aligned} \Pi_1 &= \Sigma B_n E_n(kr)\, r^n P_n(\mu)\, e^{ikCt}, \\ \Pi' &= \Sigma B'_n \psi_n(k'r)\, r^n P_n(\mu)\, e^{ikCt}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where B_n, B'_n are constants to be determined by help of the boundary conditions, $P_n(\mu)$ denotes LEGENDRE'S n^{th} coefficient, or the zonal surface harmonic of degree n.

the summation refers to integral values of n, and E_n and ψ_n denote the functions determined by the equations

$$\left.\begin{aligned} E_n(z) &= \left(\frac{1}{z}\frac{d}{dz}\right)^n \frac{e^{-iz}}{z} = 2^{1/2}\pi^{-1/2}e^{1/2(-n+1/2)i\pi}z^{-(n+1/2)}K_{n+1/2}(iz), \\ \psi_n(z) &= \left(\frac{1}{z}\frac{d}{dz}\right)^n \frac{\sin z}{z} = 2^{-1/2}\pi^{1/2}e^{-ni\pi}z^{-(n+1/2)}J_{n+1/2}(z). \end{aligned}\right\} \quad (19)$$

Here

$$K_{n+1/2}(iz) = \frac{\pi}{2\cos n\pi}\{e^{-1/2(n+1/2)i\pi}J_{-n-1/2}(z) - e^{1/2(n+1/2)i\pi}J_{n+1/2}(z)\}. \quad (20)$$

The function Π_0 can be expanded in a series of similar form. In the region $r < r_0$ the series is known (*cf.* MACDONALD (7)) to be

$$\Pi_0 = \sum_{n=0}^{\infty}(2n+1)k^{2n+1}r_0^{\,n}E_n(kr_0)r^n\psi_n(kr)P_n(\mu)e^{ikCt}, \quad (21)$$

and in the same region we have the known result that

$$\rho\frac{\partial\Pi_0}{\partial\rho} = -\sum_{n=1}^{\infty}(2n+1)k^{2n+1}r_0^{\,n-1}E_n(kr_0)r^{n+1}\psi_n(kr)(1-\mu^2)\frac{dP_n}{d\mu}e^{ikCt}. \quad (22)$$

The proof of this result involves the known equations

$$z\frac{d\psi_n(z)}{dz} = z^2\psi_{n+1}(z) = -\psi_{n-1}(z) - (2n+1)\psi_n(z), \quad (23)$$

which hold also when ψ is replaced by E, and the equations

$$(2n+1)\mu\frac{dP_n}{d\mu} = n\frac{dP_{n+1}}{d\mu} + (n+1)\frac{dP_{n-1}}{d\mu}, \qquad (2n+1)P_n = \frac{dP_{n+1}}{d\mu} - \frac{dP_{n-1}}{d\mu}. \quad (24)$$

In like manner we may obtain the equations

$$\left.\begin{aligned} \rho\frac{\partial\Pi_1}{\partial\rho} &= -\sum_{n=1}^{\infty}(2n+1)k^{2n+1}r_0^{\,n-1}E_n(kr_0)C_n r^{n+1}E_n(kr)(1-\mu^2)\frac{dP_n}{d\mu}e^{ikCt}, \\ \rho\frac{\partial\Pi'}{\partial\rho} &= -\sum_{n=1}^{\infty}(2n+1)k^{2n+1}r_0^{\,n-1}E_n(kr_0)C'_n r^{n+1}\psi_n(k'r)(1-\mu^2)\frac{dP_n}{d\mu}e^{ikCt}, \end{aligned}\right\} \quad (25)$$

where C_n and C'_n are new constants, determined in terms of the constants of types B_n and B'_n by the equations

$$\left.\begin{aligned} \frac{B_{n-1}}{2n-1}k^2 + \frac{B_{n+1}}{2n+3} &= -(2n+1)k^{2n+1}r_0^{\,n-1}E_n(kr_0)C_n, \\ \frac{B'_{n-1}}{2n-1}k'^2 + \frac{B'_{n+1}}{2n+3} &= -(2n+1)k^{2n+1}r_0^{\,n-1}E_n(kr_0)C'_n. \end{aligned}\right\} \quad (26)$$

The boundary conditions now yield, for the determination of the constants C_n, C'_n, the equations

$$\left.\begin{aligned} k^2\{\psi_n(ka)+C_nE_n(ka)\} &= k'^2C'_n\psi_n(k'a), \\ \frac{\partial}{\partial a}\{a^{n+1}\psi_n(ka)+C_na^{n+1}E_n(ka)\} &= C'_n\frac{\partial}{\partial a}\{a^{n+1}\psi_n(k'a)\}. \end{aligned}\right\} \quad . \quad . \quad . \quad (27)$$

In particular we find

$$C_n\left[\psi_n(k'a)\frac{\partial}{\partial a}\{a^{n+1}E_n(ka)\}-\frac{k^2}{k'^2}E_n(ka)\frac{\partial}{\partial a}\{a^{n+1}\psi_n(k'a)\}\right]$$
$$= -\psi_n(k'a)\frac{\partial}{\partial a}\{a^{n+1}\psi_n(ka)\}+\frac{k^2}{k'^2}\psi_n(ka)\frac{\partial}{\partial a}\{a^{n+1}\psi_n(k'a)\}. \quad . \quad . \quad (28)$$

7. We wish to evaluate the components of electric and magnetic force at points in the dielectric which are close to the sphere. We see that at such points we may put

$$\rho\frac{\partial\Pi}{\partial\rho} = -\sum_{n=1}^{\infty}(2n+1)k^{2n+1}r_0^{\,n-1}E_n(kr_0)a^{n+1}\{\psi_n(ka)+C_nE_n(ka)\}(1-\mu^2)\frac{dP_n}{d\mu}e^{ikCt}, \quad (29)$$

where

$$\psi_n(ka)+C_nE_n(ka) = \frac{\psi_n(ka)\dfrac{\partial}{\partial a}\{a^{n+1}E_n(ka)\}-E_n(ka)\dfrac{\partial}{\partial a}\{a^{n+1}\psi_n(ka)\}}{\dfrac{\partial}{\partial a}\{a^{n+1}E_n(ka)\}}$$
$$\times\left[1-\frac{k^2E_n(ka)\dfrac{\partial}{\partial a}\{a^{n+1}\psi_n(k'a)\}}{k'^2\psi_n(k'a)\dfrac{\partial}{\partial a}\{a^{n+1}E_n(ka)\}}\right]^{-1} \quad . \quad . \quad . \quad . \quad . \quad (30)$$

If the material of the sphere were perfectly conducting the second factor on the right would be replaced by unity. For any good conductor and high frequency $|k'|$ is large compared with k, and we may approximate to the value of the second factor. Put

$$k'^2 = -im^2, \qquad m^2 = 4\pi\sigma kC, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

then we may take

$$k'a = \tfrac{1}{2}ma\sqrt{2}(1-i), \qquad ik'a = \tfrac{1}{2}ma\sqrt{2}(1+i),$$

and, if a is the radius of the earth, ma is a large number, so that the most important part of $\sin k'a$ is $-\frac{1}{2}ie^{ik'a}$. Hence the most important part of $\psi_n(k'a)$ is the most important term of

$$\left\{\frac{1}{k'a}\frac{d}{d(k'a)}\right\}^n\frac{e^{ik'a}}{2ik'a},$$

and this is

$$\frac{i^ne^{ik'a}}{2i(k'a)^{n+1}}.$$

In like manner the most important part of

$$\frac{\partial}{\partial a}\{a^{n+1}\psi_n(k'a)\}$$

is

$$k'a^{n+1}\frac{i^{n+1}e^{ik'a}}{2i(k'a)^{n+1}}.$$

Thus we have the exact result that, as ma tends to become infinite,

$$\frac{1}{k'a^{n+1}\psi_n(k'a)}\frac{\partial}{\partial a}\{a^{n+1}\psi_n(k'a)\}$$

tends to i as a limit. Hence the second factor of the right-hand member of (30) may be replaced approximately by

$$1+\frac{ik^2a^{n+1}\mathrm{E}_n(ka)}{k'\frac{\partial}{\partial a}\{a^{n+1}\mathrm{E}_n(ka)\}} \qquad (32)$$

The right-hand member of (29) then takes the form of two series, of which the first is that which would occur alone if there were no resistance, and the second represents the correction for resistance.

8. The solution in the form of a series for the effect of curvature without resistance was found as early as 1903 by MACDONALD(1). He proceeded to sum the series approximately by substituting approximate values for the BESSEL'S functions which occur in the definition of the functions ψ_n and E_n. The amount of diffraction which he thus found was so great that his result was challenged by Lord RAYLEIGH(2) and POINCARÉ(3). In particular Lord RAYLEIGH pointed out that the terms of the series which would contribute most to the result would be those for which n is large of the order ka. MACDONALD(4) at once admitted the justice of the criticisms, and revised his calculations so far as to give an independent proof that the amount of diffraction at a considerable angular distance from the originating doublet must be very small. He held then, and has always maintained, that the effect of resistance would amount to no more than a practically unimportant correction.

9. With a view to the approximate summation of the series it is convenient to introduce the notation

$$\left.\begin{aligned}(-)^n z^{n+1}\mathrm{E}_n(z) &= \sqrt{\mathrm{R}_n}\,.\,e^{-i\phi_n} = v_n - iu_n,\\ (-)^n z^{n+1}\psi_n(z) &= \sqrt{\mathrm{R}_n}\,.\sin\phi_n = u_n,\\ -\tfrac{1}{2}\frac{d\mathrm{R}_n}{dz} &= \tan\chi_n.\end{aligned}\right\} \qquad (33)$$

If we identify z with ka, we may write $\mathrm{R}_{n,0}$, $\phi_{n,0}$, for the corresponding functions of kr_0. It is easy to prove the identities

$$\mathrm{R}_n\frac{d\phi_n}{dz} = v_n\frac{du_n}{dz} - u_n\frac{dv_n}{dz} = 1, \qquad (34)$$

and we find

$$(ka)^{n+1}\frac{\psi_n(ka)\frac{\partial}{\partial a}\{a^{n+1}\mathrm{E}_n(ka)\}-\mathrm{E}_n(ka)\frac{\partial}{\partial a}\{a^{n+1}\psi_n(ka)\}}{\frac{\partial}{\partial a}\{a^{n+1}\mathrm{E}_n(ka)\}}$$

$$=(-)^{n+1}i\sqrt{\mathrm{R}_n}\,.\cos\chi_n e^{i(\phi_n+\chi_n)} \quad . \quad . \quad (35)$$

and

$$\frac{ka^{n+1}\mathrm{E}_n(ka)}{\frac{\partial}{\partial a}\{a^{n+1}\mathrm{E}_n(ka)\}}=i\mathrm{R}_n\cos\chi_n e^{i\chi_n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

Hence we have at $r=a$,

$$\rho\frac{\partial\Pi}{\partial\rho}=\frac{ie^{ikCt}}{kr_0^{\,2}}\sum_{n=1}^{\infty}(2n+1)\sqrt{\frac{\mathrm{R}_{n,0}}{\mathrm{R}_n}}\,.\,e^{i(\phi_n-\phi_{n,0})}\mathrm{R}_n\cos\chi_n e^{i\chi_n}(1-\mu^2)\frac{d\mathrm{P}_n}{d\mu}$$

$$-\frac{ie^{ikCt}}{k'r_0^{\,2}}\sum_{n=1}^{\infty}(2n+1)\sqrt{\frac{\mathrm{R}_{n,0}}{\mathrm{R}_n}}\,.\,e^{i(\phi_n-\phi_{n,0})}\mathrm{R}_n^{\,2}\cos^2\chi_n e^{2i\chi_n}(1-\mu^2)\frac{d\mathrm{P}_n}{d\mu}, \quad . \quad . \quad (37)$$

where the first line of the right-hand member represents the effect of curvature without resistance, and the second line represents the correction for resistance. In practically interesting cases, z is a large number of the order 10^4 or higher, and the factor r_0^{-2} may be replaced by a^{-2}.

Approximate expressions for R_n, ϕ_n, χ_n, valid in different ranges of values of n, have been obtained after L. LORENZ* by H. M. MACDONALD(7) and J. W. NICHOLSON(9). It appears that, if r_0-a is very small compared with a, $\phi_{n,0}$ differs very little from ϕ_n throughout the whole range, and $\mathrm{R}_{n,0}/\mathrm{R}_n$ is very nearly unity until n exceeds z by a large number, but when $n-z$ is very great this fraction becomes very small. As n increases from 1, R_n increases from a number which differs very little from 1, and ultimately becomes very great, while χ_n increases continually from a very small value to $\frac{1}{2}\pi$, passing from a value slightly less than $\frac{1}{6}\pi$ to one slightly greater as n passes through the integer nearest to $z-\frac{1}{2}$. For large values of n, provided θ is not near to 0 or π, we may use the approximate formulæ

$$\left.\begin{aligned}\mathrm{P}_n&=\sqrt{\frac{2}{n\pi\sin\theta}}\,.\cos\left\{(n+\tfrac{1}{2})\theta-\frac{\pi}{4}\right\},\\(1-\mu^2)\frac{d\mathrm{P}_n}{d\mu}&=\sqrt{\frac{2\sin\theta}{n\pi}}\,.\,(n+\tfrac{1}{2})\sin\left\{(n+\tfrac{1}{2})\theta-\frac{\pi}{4}\right\},\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (38)$$

so that this factor is the product of a large number of the order $\sqrt{n}$ and a simple harmonic function of n. For values of n which are comparable with z, R_n is quite moderate. These considerations suggest that the most important terms of the series

* L. LORENZ, 'Œuvres Scientifiques,' vol. 1 (Copenhagen, 1898), p. 405.

are those for which n does not differ too much from z, and that, in summing these, the factors $\sqrt{R_{n,0}}/\sqrt{R_n}$ and $e^{i(\phi_n-\phi_{n,0})}$ may be omitted.

10. For the series, which represents the effect of curvature without resistance, methods of summation have been devised by MACDONALD(7), POINCARÉ(8), and NICHOLSON(9); and still another method has been devised by MACDONALD(11). All these methods depend upon a transformation of the series into a definite integral, and an approximate evaluation of the integral. POINCARÉ(8) did not press his method so far as to tabulate numerical results, but concluded that the expression for the electric force normal to the surface, at an angular distance θ from the originating doublet, should contain a factor of the form $e^{-A\theta}$. NICHOLSON(9) went further in the same direction, obtained a formula for the magnetic force containing such an exponential as a factor, and deduced definite numerical results. MACDONALD(7) also obtained definite numerical results which cannot be reconciled with those of NICHOLSON(9). The discrepancy was discussed by NICHOLSON(10), who traced it to an alleged flaw in the analysis used by MACDONALD(7), and it was discussed also by MACDONALD(11), who pointed out a difficulty in the analysis used by POINCARÉ(8) and NICHOLSON(9). Fresh numerical results were deduced by MACDONALD(11) from a new method of summing the series, but they do not agree with those found by NICHOLSON(9), or with those previously found by MACDONALD(7) himself.

11. It appears to be desirable to devise a new method of summing the series, to apply it to obtain definite numerical results in regard to the problem of the perfect conductor, and to compare these with the results obtained by MACDONALD and NICHOLSON, further to apply it also to calculate the effect of resistance, and finally to compare the results with those which have been found by experiment. The method which I have used is to compute a sufficient number of terms of the series and add them together. It appeared to be best, in the first instance, to do the work for a particular wave-length, and even with this limitation considerable labour was required. The quantity chosen for calculation was the magnetic force H at a point on the bounding surface.

According to (15), (37), and (38), and the results already cited as known in regard to the behaviour of the functions R_n and χ_n, we have

$$H = \frac{-ik^2}{a\sqrt{z}}\sqrt{\frac{2}{\pi\sin\theta}}\cdot\Sigma\left[\frac{(n+\frac{1}{2})^2}{z^2}\sqrt{\frac{z}{n}}\cdot R_n\cos\chi_n e^{i(kCt+\chi_n)}\left\{e^{i(z\theta+q_n\theta-\frac{1}{4}\pi)}-e^{-i(z\theta+q_n\theta-\frac{1}{4}\pi)}\right\}\right]$$
$$+\frac{ik^3}{k'a\sqrt{z}}\sqrt{\frac{2}{\pi\sin\theta}}\cdot\Sigma\left[\frac{(n+\frac{1}{2})^2}{z^2}\sqrt{\frac{z}{n}}\cdot R_n^2\cos^2\chi_n e^{i(kCt+2\chi_n)}\left\{e^{i(z\theta+q_n\theta-\frac{1}{4}\pi)}-e^{-i(z\theta+q_n\theta-\frac{1}{4}\pi)}\right\}\right], \quad . \quad . \quad (39)$$

where the summation now refers to the relevant terms, and

$$q_n = n+\tfrac{1}{2}-z. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

This equation (39) may be expressed in a real form. Corresponding to primary waves given by

$$\Pi_0 = \frac{\cos k(Ct-R)}{R}, \qquad (41)$$

we find

$$\begin{aligned} H = &-\frac{k^2}{a\sqrt{z}}\sqrt{\frac{2}{\pi\sin\theta}}\,.\,[(S_{11}+S_{12})\sin(kCt-z\theta)-(S_{21}-S_{22})\cos(kCt-z\theta) \\ &\qquad -(S_{11}-S_{12})\sin(kCt+z\theta)-(S_{21}+S_{22})\cos(kCt+z\theta)] \\ &+\frac{k^3}{ma\sqrt{z}}\sqrt{\frac{2}{\pi\sin\theta}}\,.\,[(S'_{11}-S'_{12})\sin(kCt-z\theta)+(S'_{21}+S'_{22})\cos(kCt-z\theta) \\ &\qquad -(S'_{21}-S'_{22})\sin(kCt+z\theta)-(S'_{11}+S'_{12})\cos(kCt+z\theta)], \end{aligned} \qquad (42)$$

where $m = \sqrt{(4\pi\sigma kC)}$, and

$$\left.\begin{aligned} S_{11} &= \Sigma\frac{(n+\frac{1}{2})^2}{z^2}\sqrt{\frac{z}{n}}\,.\,R_n\cos^2\chi_n\cos\left(q_n\theta-\frac{\pi}{4}\right), \\ S_{12} &= \Sigma\frac{(n+\frac{1}{2})^2}{z^2}\sqrt{\frac{z}{n}}\,.\,R_n\cos^2\chi_n\tan\chi_n\sin\left(q_n\theta-\frac{\pi}{4}\right), \\ S_{21} &= \Sigma\frac{(n+\frac{1}{2})^2}{z^2}\sqrt{\frac{z}{n}}\,.\,R_n\cos^2\chi_n\sin\left(q_n\theta-\frac{\pi}{4}\right), \\ S_{22} &= \Sigma\frac{(n+\frac{1}{2})^2}{z^2}\sqrt{\frac{z}{n}}\,.\,R_n\cos^2\chi_n\tan\chi_n\cos\left(q_n\theta-\frac{\pi}{4}\right), \end{aligned}\right\} \qquad (43)$$

and

$$\left.\begin{aligned} S'_{11} &= \Sigma\frac{(n+\frac{1}{2})^2}{z^2}\sqrt{\frac{z}{n}}\,.\,R_n^2\cos^4\chi_n(1-\tan^2\chi_n)\sin q_n\theta, \\ S'_{12} &= \Sigma\frac{(n+\frac{1}{2})^2}{z^2}\sqrt{\frac{z}{n}}\,.\,R_n^2\cos^4\chi_n\,2\tan\chi_n\cos q_n\theta, \\ S'_{21} &= \Sigma\frac{(n+\frac{1}{2})^2}{z^2}\sqrt{\frac{z}{n}}\,.\,R_n^2\cos^4\chi_n(1-\tan^2\chi_n)\cos q_n\theta, \\ S'_{22} &= \Sigma\frac{(n+\frac{1}{2})^2}{z^2}\sqrt{\frac{z}{n}}\,.\,R_n^2\cos^4\chi_n\,2\tan\chi_n\sin q_n\theta. \end{aligned}\right\} \qquad (44)$$

12. It may be observed here that similar expressions can be obtained for the values of E_r and E_θ, the radial and tangential components of electric force at a point on the surface. With the formula (39) for H we should find $E_\theta = kH/k'$, where the second line of the expression for H may be omitted. This form gives the amplitude of E_θ at any distance as k/m times the amplitude of H at the same distance. In the case of primary waves given by (41) the value of E_r can be obtained from the formula (42) for H by changing the signs of the coefficients of $\sin(kCt-z\theta)$ and $\cos(kCt-z\theta)$, and in the sums such as S_{11} and S'_{11} replacing $(n+\frac{1}{2})^2 n^{-1/2} z^{-3/2}$ by $(n+\frac{1}{2})(n+1)n^{1/2}z^{-5/2}$. Since z is a large number, and, in the relevant terms, n differs but little from z, it is manifest that the terms of E_r which contain the sine or cosine of $(kCt-z\theta)$, are nearly the same,

except for a change in sign, as the corresponding terms of H, while those which contain the sine or cosine of $(kCt+z\theta)$ are nearly the same and of the same sign.

13. In order to sum the series denoted by S_{11}, ... it was necessary to evaluate R_n and $\tan\chi_n$. This can be done, when u_n and v_n are known, by means of the formulæ

$$\left.\begin{aligned} R_n &= u_n^2+v_n^2, \\ \tan\chi_n &= u_n u_{n+1}+v_n v_{n+1}-\frac{n+1}{z}(u_n^2+v_n^2), \end{aligned}\right\} \quad . \quad . \quad . \quad (45)$$

the second of these being obtained from (23). It was therefore necessary to calculate u_n and v_n. Now, when q_n is not more than a small fraction of $(\frac{1}{6}z)^{1/3}$, it is known from the analysis of L. LORENZ already cited, and confirmed by MACDONALD and NICHOLSON, that v_n and u_n are given very approximately by the equations

$$\left.\begin{aligned} v_n &= 2^{5/6}\,3^{-2/3}\pi^{-1/2}z^{1/6}\cos^2\tfrac{1}{6}\pi\,\{\Gamma(\tfrac{1}{3})+(\tfrac{1}{6}z)^{-1/3}q_n\,\Gamma(\tfrac{2}{3})\}, \\ u_n &= 2^{5/6}\,3^{-2/3}\pi^{-1/2}z^{1/6}\cos\tfrac{1}{6}\pi\,\sin\tfrac{1}{6}\pi\,\{\Gamma(\tfrac{1}{3})-(\tfrac{1}{6}z)^{-1/3}q_n\,\Gamma(\tfrac{2}{3})\}. \end{aligned}\right\} \quad . \quad . \quad . \quad (46)$$

These results were used for the values $-\frac{1}{2}$ and $\frac{1}{2}$ of q_n, corresponding to the values z and $z-1$ of n, but not for any numerically larger values of q_n. For other values of n the values of u_n and v_n were found from the sequence equations, deduced from (23), viz. :—

$$v_n+v_{n-2}=\frac{2n-1}{z}v_{n-1}, \qquad u_n+u_{n-2}=\frac{2n-1}{z}u_{n-1}. \quad . \quad . \quad . \quad . \quad (47)$$

For n increasing beyond z $(q_n>\frac{1}{2})$, these were used in the equivalent forms

$$v_n=2v_{n-1}-v_{n-2}+\frac{2(q_n-1)}{z}v_{n-1}, \qquad u_n=2u_{n-1}-u_{n-2}+\frac{2(q_n-1)}{z}u_{n-1}, \quad . \quad (48)$$

and, for n decreasing beyond $z-1$, $(q_n<-\frac{1}{2})$, in the forms

$$v_n=2v_{n+1}-v_{n+2}-\frac{2(-q_n-1)}{z}v_{n+1}, \qquad u_n=2u_{n+1}-u_{n+2}-\frac{2(-q_n-1)}{z}u_{n+1}, \quad (49)$$

and the results were verified each time by substituting in the formulæ (47).

The wave-length taken was 5 km., so that ka, or z, was 8000.

The initial results thus found are given in the following table :—

TABLE I.

q_n.	u_n.	v_n.	R_n.	$\tan\chi_n$.
$-\frac{1}{2}$	2·56472	4·24281	24·5792	0·5508
$\frac{1}{2}$	2·44958	4·44223	25·7338	0·6040

It seems to be unnecessary to print a complete table of the results. Six figures were retained in computing u_n, v_n, R_n, and this made it possible to compute $\tan \chi_n$ to four figures, as above. The computation was carried from $q_n = -\frac{135}{2}$ to $q_n = \frac{135}{2}$.

The corresponding values of

$$\text{(I.)}\ \frac{(n+\frac{1}{2})^2}{z^2}\sqrt{\frac{z}{n}}\,.\,R_n \cos^2 \chi_n, \qquad \text{(II.)}\ \frac{(n+\frac{1}{2})^2}{z^2}\sqrt{\frac{z}{n}}\,.\,R_n \cos^2 \chi_n \tan \chi_n,$$

$$\text{(III.)}\ \frac{(n+\frac{1}{2})^2}{z^2}\sqrt{\frac{z}{n}}\,.\,R_n^2 \cos^4 \chi_n (1-\tan^2 \chi_n), \qquad \text{(IV.)}\ \frac{(n+\frac{1}{2})^2}{z^2}\sqrt{\frac{z}{n}}\,.\,R_n^2 \cos^4 \chi_n\, 2 \tan \chi_n,$$

were then computed for the same values of q_n, four figures being retained. The general character of the results can be exhibited to the eye by marking on squared paper the points which have the above expressions as ordinates and the corresponding values of $n-z$, or $n-8000$, as abscissæ, and drawing smooth curves through the points. The four curves, numbered I., II., III., IV., in the same order as the four expressions, are shown in fig. 1, p. 120. In order to keep the figure within bounds, the values of the expressions III. and IV. have been divided by 10 before marking the points representing them.

14. The next thing to be done was to sum the series denoted by S_{11}, ... for a number of values of θ. The values chosen were $\pi/30$, $\pi/20$, $\pi/15$, $\pi/12$, $\pi/10$ or 6°, 9°, 12°, 15°, 18°. Every term of each of these series is the product of a number shown by an ordinate of one of the curves I., ... IV., the ordinates of the curves III., IV. being multiplied by 10, and a simple harmonic function of n, actually a sine or cosine of

$$\theta(n-8000)+\tfrac{1}{2}\theta-\tfrac{1}{4}\pi \quad \text{or} \quad \theta(n-8000)+\tfrac{1}{2}\theta.$$

Thus the terms of the series can be represented graphically by ordinates corresponding to integral values of $n-8000$ as abscissæ, and a smooth curve drawn through the extremities of the ordinates. The curve fluctuates like a curve of sines, but the amplitude is variable, and the terms may be divided into groups of one sign by means of the points in which the curve cuts the axis of abscissæ, and these again into halved groups by means of the abscissæ for which the corresponding sine or cosine is a maximum or a minimum. The portions of the curves that correspond to these groups and halved groups of terms may be described as "bays" and "half-bays." The method adopted for summing the series is analogous to the method employed by FRESNEL in problems of diffraction, wherein use was made of the notion of "half-period elements"; when the amplitude is nearly constant the terms belonging to a bay are nearly cancelled by those belonging to the two neighbouring half-bays.

It seems best to illustrate the method by explaining how S_{12} was evaluated for $\theta = 12°$. A portion of the fluctuating curve, and the corresponding portion of the

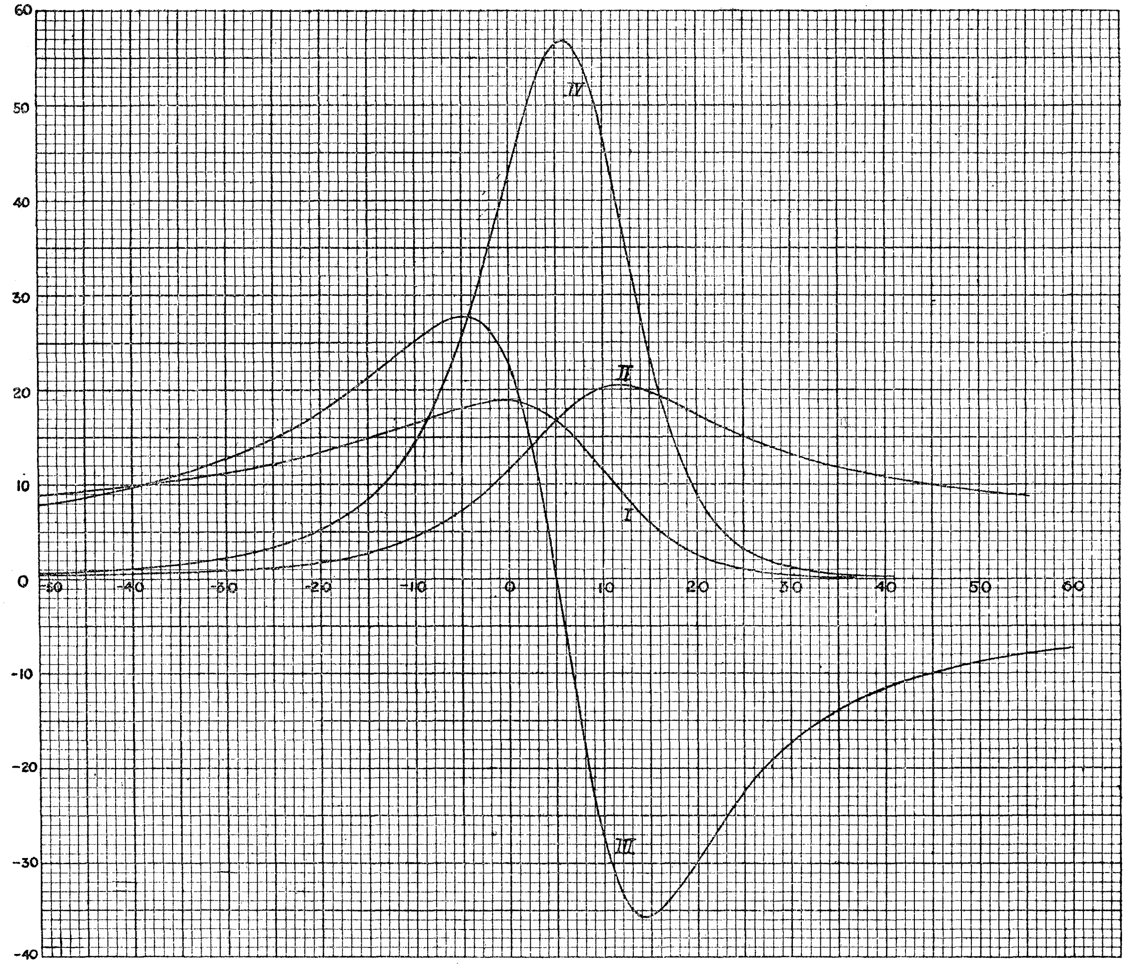

Fig. 1.

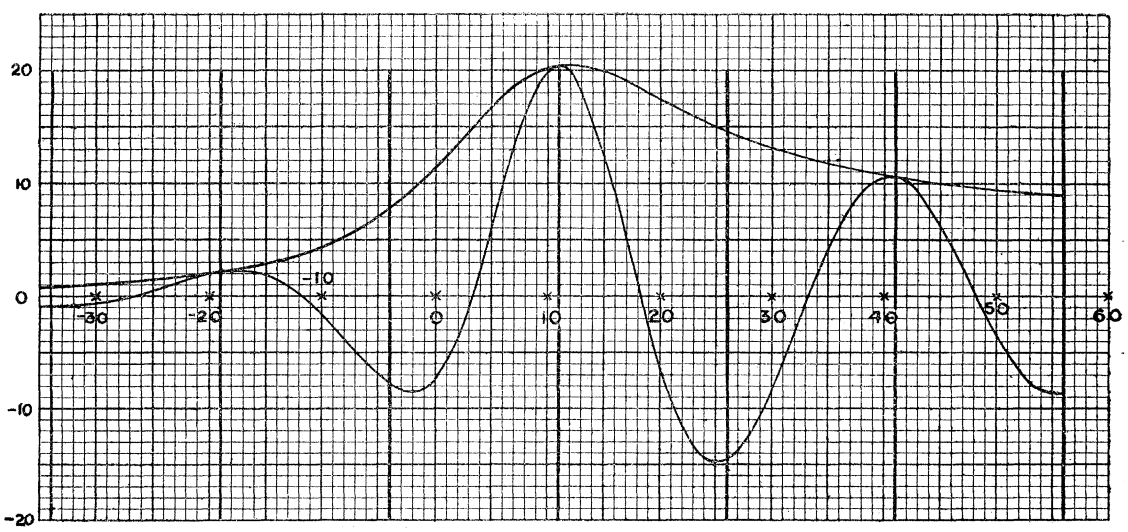

Fig. 2.

amplitude curve (II.), are shown in fig. 2, in which the ordinates that separate the half-bays are also drawn. The greatest ordinates are in the bay between the values 4 and 18 of $n-8000$; with this were taken the two half-bays from -4 to 3 and from 19 to 26. The next similar portion on the left consists of the half-bay from -34 to -27, the bay from -26 to -12, and the half-bay from -11 to -4; and the next similar piece on the right consists of the half-bay from 26 to 33, the bay from 34 to 48, and the half-bay from 49 to 56. The ordinates common to two consecutive half-bays were halved, and half taken as contributed by each. The sums of the terms thus contributed to the series by the three pieces of the curve described above were approximately 61, -14, and -5, which yield as sum 42. It may also be seen that this is very nearly the sum of the whole series. After the critical middle piece (-4 to 26) the series may be broken up into sums of terms contributed by pairs of consecutive half-bays, *e.g.*, 26 to 33 and 34 to 41, are such a pair, 41 to 48 and 49 to 56 are the next pair, and so on. The sums contributed by such pairs have alternate signs and diminish continually in absolute magnitude. It thus becomes easy to estimate the error made in breaking off the sum at specified maxima, or minima, of the sine or cosine involved. In the particular example of S_{12} for $\theta = 12^\circ$, the error made in breaking off the sum at -34 and 56 was estimated in this way as about -1; and the value of S_{12} for $\theta = 12^\circ$ was taken to be about 41.

15. The value of H expressed by (42) involves eight of these series, but the labour of calculation can be reduced very much by observing the forms of the two expressions in square brackets in (42). The terms written in the first lines of these expressions represent waves travelling outwards from the originating doublet, with a velocity C along the arc of a great circle drawn from the originating doublet to the point of observation, the wave having at each point an amplitude and phase depending upon the values of $S_{11}, \ldots$ at the point. The second lines represent a return wave. Now it is evident that for moderate values of θ, say less than $\frac{1}{2}\pi$, the amplitude of the return wave must be very small, and thus we are led to expect that, at least approximately,

$$S_{11} = S_{12}, \qquad S_{21} = -S_{22}, \qquad S'_{11} = -S'_{12}, \qquad S'_{21} = S'_{22}. \quad . \quad . \quad . \quad (50)$$

An analytical proof that the return wave is negligible in the case of perfect conduction has been given by MACDONALD(7), and the above approximate relations have been verified numerically in a few cases. By assuming them throughout the labour of calculation could have been considerably reduced.

16. The values computed for S_{12}, S_{22}, S'_{12}, S'_{22} for the selected values of θ are recorded in the following table:—

TABLE II.

θ.	S_{12}.	S_{22}.	S'_{12}.	S'_{22}.
6°	102	150	6197	3540
9°	73	44	2830	3540
12°	41	6	749	2604
15°	19	−6·5	−182	1550
18°	7·5	−6·5	−456	760

The amplitude of H may now be taken to be

$$\frac{2\sqrt{2}\,.\,k^2}{a\sqrt{(\pi z \sin\theta)}}\sqrt{\left\{\left(S_{12}+\frac{k}{m}S'_{12}\right)^2+\left(S_{22}-\frac{k}{m}S'_{22}\right)^2\right\}} \quad . \quad . \quad . \quad . \quad (51)$$

and the value of this quantity for any particular value of θ may be compared with the amplitude of H_0, the magnetic force at the same point due to the originating doublet alone. Now, when θ is not nearly equal to 0, R is large of the same order as a, and we have the approximate equation

$$H_0 = -\frac{k^2 \sin\theta}{4a\sin^2\frac{1}{2}\theta}\cos k(Ct-R), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

so that we may take the ratio of amplitudes to be

$$\frac{2\sqrt{(2\sin\theta)}}{\sqrt{(\pi z)}\,.\cos^2\frac{1}{2}\theta}\sqrt{\left\{\left(S_{12}+\frac{k}{m}S'_{12}\right)^2+\left(S_{22}-\frac{k}{m}S'_{22}\right)^2\right\}} \quad . \quad . \quad . \quad . \quad (53)$$

The ratio of amplitudes is the quantity tabulated by MACDONALD(11) in the case of perfect conduction. It has been computed anew by the method described above for the wave-length 5 km. and two values of the ratio k/m, viz., the value zero, answering to the hypothesis of perfect conduction, and the value 0·001826 answering to the conductivity of sea-water ($\sigma = 10^{-11}$). In the following table the first column gives the value of θ in degrees; the second the corresponding arc-distance D in kilometres; the third headed A_M, the amplitude ratio as computed by MACDONALD(11); the fourth, headed A_∞, the amplitude ratio computed by means of the formula (53) when k/m is taken to be zero; the fifth, headed A_σ, the amplitude ratio computed by means of the same formula when k/m is taken to be 0·001826. There is some uncertainty about the third figure in the 6° line in each of the last three columns.

TABLE III.

θ.	D.	A_M.	A_∞.	A_σ.
6°	667	1·04	1·05	1·05
9°	1000	0·61	0·61	0·61
12°	1333	0·34	0·34	0·35
15°	1667	0·18	0·18	0·19
18°	2000	0·10	0·10	0·11

Except as regards the uncertain last figures in the 6° line there is complete agreement between the third and fourth columns of the table; in other words, my results confirm those of MACDONALD(11) for the case of perfect conduction. The methods of summation employed by him are so different from those which I have adopted that the results may be accepted with great confidence. The correction for resistance seems to be rather larger than MACDONALD anticipated, but nevertheless not large enough to have the importance expected by SOMMERFELD, although his expectation that it would be increased by curvature is verified.

17. Since MACDONALD's result for the problem of the perfect conductor is confirmed, it appears to be unnecessary to enter into a detailed comparison with the results obtained by POINCARÉ(8) and NICHOLSON(9), although it may be stated that the objection raised by MACDONALD(11) to a step in their analysis seems to the present writer to be well founded. But it is appropriate here to notice a solution of the problem put forward by MARCH(12) and RYBCZYŃSKI(13). MARCH, apparently acting upon a suggestion of SOMMERFELD's (see "Report" (17)), set out to obtain a solution of the problem in terms of definite integrals, analogous to that obtained by SOMMERFELD(6) for the problem of the plane boundary. He confined his analysis to the case of a perfect conductor; and his final result was that the amplitude of the waves, received at an angular distance θ from the originating doublet, should be very approximately proportional to $(\theta \sin \theta)^{-1/2}$. An oversight in his work was pointed out by POINCARÉ,* and its correction was undertaken by RYBCZYŃSKI(13) who also obtained a result differing from that of MACDONALD(11); but MACDONALD(11) has

* H. POINCARÉ, 'Paris C.R.,' vol. 154 (1912), p. 795.

noted a difficulty in RYBCZYŃSKI's analysis. It seem to me, however, that there is a serious error in the part of MARCH's work which is accepted by RYBCZYŃSKI. He set out to determine a function, say ψ, which has the property expressed by the equation

$$H_\phi = \frac{\partial \psi}{\partial \theta}.$$

The function ψ satisfies the same differential equation as the Hertzian function Π, the same conditions of continuity, the same condition at infinite distances, and similar boundary conditions; and the special form of it answering to the originating doublet, say ψ_0, is given by the equation

$$\psi_0 = \frac{e^{ik(Ct-R)}}{R}.$$

In the region outside the sphere ψ has the form $\psi_0+\psi_1$, where ψ_1 in the analogue of Π_1, in the region inside the sphere it may be denoted by ψ', analogous to Π'. Now MARCH seeks a solution in which ψ_1 and ψ' have the forms

$$\psi_1 = \frac{1}{r_0}\int_{-1/2}^{\infty} (\alpha+\tfrac{1}{2})\, P_\alpha(\cos\theta)\, r^\alpha E_\alpha(kr) e^{ikCt} F_1(\alpha)\, d\alpha,$$

$$\psi' = \frac{1}{r_0}\int_{-1/2}^{\infty} (\alpha+\tfrac{1}{2})\, P_\alpha(\cos\theta)\, r^\alpha \psi_\alpha(k'r)\, e^{ikCt} F(\alpha)\, d\alpha,$$

where $P_\alpha(\cos\theta)$ denotes a certain solution of LEGENDRE's equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dP_\alpha(\cos\theta)}{d\theta}\right)+\alpha(\alpha+1)\,P_\alpha(\cos\theta)=0,$$

in which α is not in general an integer, $E_\alpha(kr)$ and $\psi_\alpha(k'r)$ are defined by means of BESSEL's functions as in equations (19), but not by means of differential operators, and $F_1(\alpha)$ and $F(\alpha)$ are functions of α to be determined. He also obtains integral formulæ of a similar type to represent ψ_0 in the regions $r>r_0$ and $r_0>r>a$. The function denoted by $P_\alpha(\cos\theta)$ is free from singularity at $\theta = 0$.

Now these forms fail to satisfy conditions which must necessarily be imposed on ψ_0, ψ_1, and ψ'. It is necessary that ψ' should be finite at $r = 0$; but when $r = 0$, and α lies between $-\frac{1}{2}$ and 0, $r^\alpha\psi_\alpha(k'r)$ is infinite. Further it is necessary that ψ_0, ψ_1, and ψ' should be free from singularity on the axis $\theta = \pi$; but when α is not an integer, and $P_\alpha(\cos\theta)$ is finite at $\theta = 0$, it is infinite at $\theta = \pi$. Hence the form taken for ψ' has a singular point at the centre of the sphere, and a line of singular points extending thence along the radius of the sphere drawn in the direction $\theta = \pi$, and the forms taken for ψ_0 and ψ_1 have a line of singular points extending along the continuation of this radius outside the sphere. It appears therefore that no solution of the type sought by MARCH exists.

This error seems to me to vitiate the whole of the work of MARCH and RYBCZYŃSKI. In particular, it seems to destroy the foundation for RYBCZYŃSKI's solution of the problem of the perfect conductor, and his extension of the solution to include the effect of resistance.

18. A result that should admit of being tested experimentally is the law of decrease of amplitude of the electromagnetic waves with increasing distance. According to what has been said in §§ 12 and 15, this law must be very nearly the same for the electric force in the field as for the magnetic, and it may be assumed that the amplitude of the received antenna current at any place is proportional to the amplitude of the magnetic force of the field at that place. The various diffraction theories lead to approximate formulæ for the law of decrease of the amplitude of the magnetic force. Let H denote this amplitude at angular distance θ, H_1 the corresponding amplitude at angular distance θ_1, and λ the wave-length, measured in kilometres. Then a result given by MACDONALD (11) leads to the formula

$$\frac{H}{H_1} = \frac{\cos \frac{1}{2}\theta}{\cos \frac{1}{2}\theta_1} \sqrt{\left(\frac{\sin \frac{1}{2}\theta_1}{\sin \frac{1}{2}\theta}\right)}\, e^{(47\cdot 89)\, \lambda^{-1/3} (\sin {}^1\!/_2\theta_1 - \sin {}^1\!/_2\theta)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

a result given by NICHOLSON (9) leads to the formula,

$$\frac{H}{H_1} = \sqrt{\left(\frac{\sin \frac{1}{2}\theta}{\sin \frac{1}{2}\theta_1}\right)}\, e^{(23\cdot 8)\, \lambda^{-1/3} (\theta_1 - \theta)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

and a result given by RYBCZYŃSKI (13) to the formula,

$$\frac{H}{H_1} = \sqrt{\left(\frac{\theta_1 \sin \theta_1}{\theta \sin \theta}\right)}\, e^{(11\cdot 3)\, \lambda^{-1/3} (\theta_1 - \theta)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

while ZENNECK (18) gives a formula equivalent to

$$\frac{H}{H_1} = \sqrt{\left(\frac{\theta_1 \sin \theta_1}{\theta \sin \theta}\right)}\, e^{(11\cdot 9)\, \lambda^{-1/3} (\theta_1 - \theta)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

as the result of a correction of the work of MARCH (12). In the exponential factors of the last three formulæ θ and θ_1 are measured in radians.

The discrepancies between these various formulæ are sufficient to justify the attempt to obtain an independent solution of the diffraction problem. The confirmation of MACDONALD's result for a particular wave-length affords good reason for accepting his formula as correct, and consequently rejecting the others as incorrect, apart from the objections which have been brought against the analytical procedure of their authors.

MACDONALD has pointed out that the range of validity of such a formula as (54) is restricted by the condition that neither θ nor θ_1 must be too small. For $\lambda = 5$ it begins to be valid at about 7°, for $\lambda = 2\cdot 56$ at about 6° 20′, for $\lambda = 1\cdot 22$ at about 6°. If it is desired to compare it with results of experiment, the distances to be considered should not be less than about 400 sea miles (6° 40′), or 750 km. (6° 45′). For the purpose of testing it experimentally it is necessary to measure the amplitudes of received antenna currents at various distances exceeding these limits.

19. So far as I know, the only records of quantitative measurement of received current at sufficiently distant places are contained in the memoir of AUSTIN(14) and the article by HOGAN(15). AUSTIN's experiments were performed by transmitting signals between the U.S. station at Brant Rock and two cruisers in the Atlantic, measurements being made of the strength of the signals from shore to ship, from ship to shore, and from ship to ship, at various distances up to 1000 sea miles (1850 km.), by day and by night, during several months in the years 1909–10. The wave-lengths employed were 3·75 km., 1·5 km., and 1 km. HOGAN's experiments were performed in the year 1913 between the U.S. station at Arlington, Va., and one of the same two cruisers. The wave-lengths employed were 3·8 km. for the signals sent from the shore station, 2 km. for those sent from the ship. The range of distance was 3000 km. for the shorter waves, and 4250 km. for the longer. The method of observation was to take shunted telephone readings on the incoming signals, the shunt being adjusted to reduce the signals to audibility, and the standard of audibility being that strength of signal which permits a clear differentiation of dots and dashes. AUSTIN's detectors were of the "free wire electrolytic type," HOGAN generally used the "Fessenden liquid barretter," but "some of the readings at extreme distances were taken upon the heterodyne receiver." HOGAN defines the "audibilty factor" as the ratio $(R+S)/S$, where R is the impedance of the telephone, and S that of the shunt which, when connected across the terminals of the telephone receiver, reduces the signal intensity to audibility. He states that this ratio is approximately proportional to the square of the received antenna current. AUSTIN records the results of his experiments of July, 1910, in tables. He also records all his results, including these, graphically, by marking on squared paper points whose abscissæ are the distances, and ordinates the received antenna currents. HOGAN records his results graphically in a similar way, with the difference that his ordinates are proportional to the values of the audibility factor. Both observers found the results for daylight signalling much more regular than for night.

20. AUSTIN sought to deduce from his observations the law of decrease of received antenna current with increasing distance. In this he was guided partly by some observations taken in 1905 in the Irish Sea by DUDDELL and TAYLOR, who, he says, found that the received current over water fell off nearly in proportion to the distance. This was to be expected for comparatively short distances. He was also guided partly by the ideas of ZENNECK(5), referred to in § 3 above. Accordingly he compared his results with an expression of the type $Ae^{-\alpha D}/D$, in which D denotes distance from the sending station, and A and α are independent of D, but may depend upon the wave-length. The factor $e^{-\alpha D}$ he described as due to "absorption." The most regular series of his observations taken in July, 1910, were found to fit fairly well the curves that

can be obtained by taking $\alpha = (0\cdot0015)\lambda^{-1/2}$, and introducing suitable values for A, the wave-length λ and the distance D being measured in kilometres. Similar curves were drawn by him on the diagrams representing all the series of his observations, and they show that there is a fair agreement between the daylight observations and the formula. HOGAN drew on his diagrams the graphs of expressions of the type $(Ae^{-\alpha D}/D)^2$, with AUSTIN's values for α in terms of wave-lengths and suitable values for A, and found that the graphs fitted the daylight observations rather well.

21. When AUSTIN's formula is expressed in terms of angular distance, so as to become comparable with those written down in § 18 above, it takes the form

$$\frac{H}{H_1} = \frac{\theta_1}{\theta}\, e^{(9\cdot6)\lambda^{-1/2}(\theta_1-\theta)} \qquad (59)$$

This formula would clearly not show, in a narrow range such as that between 400 and 1000 sea miles, very much divergence from those given by RYBCZYŃSKI(13) and ZENNECK (18); and both these writers claim that their formulæ represent the results of AUSTIN's experiments better than his own. On the other hand, it would show even in this range a well-marked divergence from MACDONALD's formula, though not perhaps sufficient to be conclusive. When the comparison is extended to the wider range covered by HOGAN's experiments, the divergence would appear to be decisive. If the records of the observations could be accepted without criticism, it could be stated that the law of decrease of amplitude of the electro-magnetic waves with increasing distance from the sending station, as expressed by the formula (59), has been tested and found adequate over a wide range of distances and wave-lengths; and further, since this formula cannot be reconciled even approximately with MACDONALD's theory, it could be inferred, as in fact it has been, that diffraction cannot account for the observed facts.

22. I hesitate to draw this inference for two reasons. First, there is some doubt as to what the observed facts really are. Second, a different interpretation of the available records leads to precisely the opposite conclusion.

As regards the observations, I wish to make it clear at the outset that I do not undervalue the work of AUSTIN and HOGAN. The investigations which they undertook were of such difficulty that to obtain results, so consistent as theirs, must have required much patient labour and an uncommon degree of skill. Any criticism that I venture to make is prompted solely by the desire to arrive at a more complete comprehension of the matter.

The method of observation by means of shunted telephone readings appears not to admit of great accuracy. AUSTIN states that the errors of such readings on board ship with good operators in good weather may amount to as much as 20 to 40 per cent. In stormy weather they are increased. There is some obscurity as to the meaning of the numbers and ordinates which represent in AUSTIN's tables and diagrams the values of the "received current." Many passages convey the impression that they are proportional to the values of the square root of HOGAN's "audibility factor," but this is not stated expressly. As the received antenna current was inferred in some way from the value of this factor, it seems unfortunate that the value of the factor was not recorded. The behaviour of the detectors used for the shunt readings is not thoroughly understood by electricians. For example, FLEMING (*op. cit. ante*, p. 397) states concerning the Fessenden liquid barretter, used by HOGAN, that its action is probably electrolytic, and due to annulment of polarization, rather than purely thermal. HOGAN does not state how the approximate law of squares connecting his audibility factor with the received current was verified, but FLEMING's statement would seem to indicate that the action of his detectors would approximate to that of the detector used by AUSTIN. Now AUSTIN has investigated the action of his detector. In a series of experiments performed at Brant Rock he measured the same antenna currents in two ways: (1) by means of the shunted telephone and detector; (2) by means of a rectifier, which was "connected in a secondary circuit coupled to the antenna, and calibrated by means of a thermo-element in the antenna and an exciting buzzer circuit which could be tuned to the wave-length used." Apparently the rectifier had to be used because it was desired to measure currents too weak to be measured by the thermo-element, but it is not clear how the rectifier was calibrated for such weak currents. The results of these experiments are recorded by him in a table (p. 319), which does not support the conclusion that the current is proportional to the square root of the audibility factor. The quantities recorded in the table are the values of the shunt [S] in ohms, and the antenna current [I] in micro-amperes, the resistance [R] of the telephone being 600 ohms. From this table I find that (R+S)/S is nearly proportional to I^2 for large values of I, nearly proportional to I for small values. This result appears from the following table compiled from AUSTIN's.

TABLE IV.

I	95	122	150	194	474	568	672
$\frac{SI^2}{R+S}$	361	363	369	373	374	376	376

I	10	13	15	18	22	26	29
$\frac{SI}{R+S}$	10	10·8	11·5	11·3	11	10·4	9·7

In view of this result, and the fact that most of the numbers recorded by AUSTIN as values of received antenna current at distances exceeding 500 sea miles are less than 29, it seemed to be worth while to see if agreement between MACDONALD'S formula (54) and observation could be attained by supposing the amplitude of the electromagnetic waves at great distances, and therefore the values of the received antenna current, to be proportional to the audibility factor instead of its square root. I therefore compared the formula in the first instance with HOGAN'S long-distance observations. The result of the comparison is shown in figs. 3 and 4, in which the abscissæ of the points marked with crosses represent distances, and the ordinates the values of the audibility factor at those distances, as observed by HOGAN by daylight. Everything in these diagrams, except the curved lines, is taken from HOGAN'S diagrams; and the curved lines are the graph of H/H_1 as given by formula (54) for wave-length 3·8 km., the scale being adjusted so that the highest point of the curve in fig. 3 answers to angular distance 9° (1000 km.). It will be seen that the graph

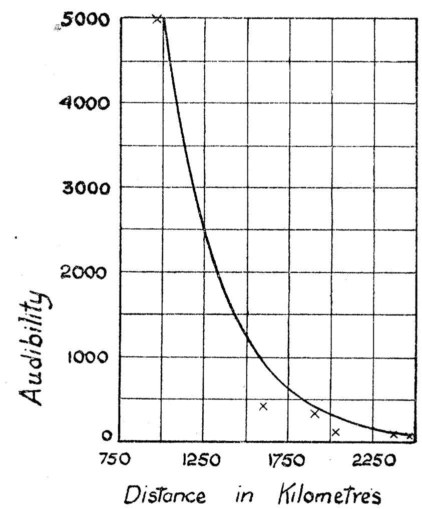

FIG 3.

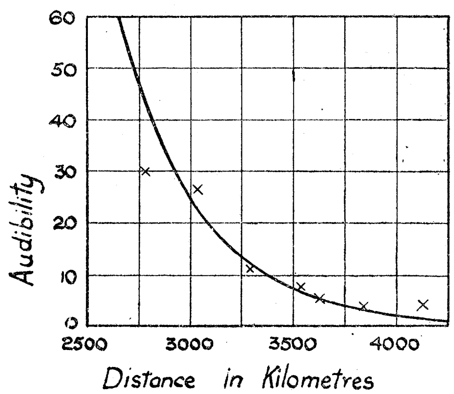

FIG 4.

fits the observations rather well. I made a similar comparison for HOGAN'S daylight observations on the wave-length 2 km., and again found a good fit. Then I compared the values of the square root of the right-hand member of formula (54) with the values recorded in AUSTIN'S diagrams for wave-lengths 1 km., 1·5 km., and 3·75 km., and in every case found that the daylight observations for distances exceeding 500 sea miles were in good agreement with the formula thus modified.

[*Note added February* 6, 1915.—After the paper was read, Prof. C. H. LEES kindly brought to my notice some more recent experiments of L. W. AUSTIN, which are recorded, at present in abstract only, in the 'Journal of the Washington Academy of Sciences,' of date December 4, 1914. In these experiments the wave-length used was 3·8 km., and observations were taken at various distances ranging from 556 km. to 3700 km. The method of experiment appears to have been the same as in

AUSTIN's earlier work. The values of received antenna current, obtained from the "smoothed curve of observations," are recorded, and compared with the results that would be obtained from the author's formula (§ 20 above), and with those that would be obtained from a formula described as the "Sommerfeld transmission formula," which is effectively equivalent to (57). I find that, in this case also, the ratios of the quantities recorded as observed currents at different distances are nearly the same as the square roots of the ratios of the magnetic forces at those distances, as calculated from MACDONALD's formula.]

23. From this critical discussion I draw the inference that there is nothing in the experimental evidence, at present available, to compel us to adopt the view that the diffraction theory fails to account for the facts. On the contrary, that evidence can be interpreted in such a way as to support the view that the results of the diffraction theory accord well with those of daylight observations. Until more complete experimental data are available the question of the success or failure of the diffraction theory must remain open.

However this question may ultimately be settled, the discussion shows that it is impossible to accept the hypothesis that the law of decrease of the forces of the electromagnetic field with increasing distance is a combination of the law of spherical divergence with a law of absorption, expressed by an attenuation factor of exponential type. However closely such a law may represent the facts it can have no value except as an empirical formula. In particular, it is not admissible to draw from it any inference as to the amount of absorption. Independently of any absorption which may exist, there must be a law of diffraction expressing attenuation of the field on account of the curvature of the earth's surface; and the type of attenuation factor required to express the effect of curvature would not differ much from the exponential type in a moderate range of great distances.

The investigations of SOMMERFELD and those of this paper throw some light on the effect of the resistance of the medium, over the surface of which signals are transmitted. It appears that, with such wave-lengths as are used in practice, a moderate amount of resistance, such as that of sea-water, increases the strength of the signals at great distances. A slightly higher resistance would increase it still more, but no conclusion can safely be drawn as to the effect of so high a resistance as that of dry ground. It seems to me, however, to be not unlikely that even a rather high resistance may be favourable, and I am inclined to regard the known fact that signals are in general appreciable at greater distances over sea than over land, as an effect of the broken surface of ground covered with rocks, buildings, or trees. This question also remains to be settled by quantitative experimental investigation.

Another difficult question presents itself in the known fact that the signals are generally stronger by night than by day, and the related fact that the attenuation of night signals by distance is sometimes less than it would be if they diminished simply according to the law of spherical divergence. These facts suggest emphatically that there is, especially at night, some other cause at work besides diffraction, and that it may be necessary to take into account the possibilities involved in transmission through a heterogeneous, and perhaps in parts conducting, atmosphere, as is maintained particularly by ECCLES(16) and (17). The view that ionization in the upper and middle atmosphere may be a cause of variations of the medium, large enough to be favourable to long-distance transmission, inasmuch as the waves may be partially refracted downwards, and liable, on occasion, to be unfavourable, inasmuch as the refracted waves may be subject to absorption, demands scrutiny, to be conducted by means of careful quantitative experiments under varying conditions and controlled by an adequate mathematical analysis. The adoption of this view at present seems to me to be premature.

It remains to acknowledge gratefully the valuable help which I have received from Prof. J. S. TOWNSEND, F.R.S., in the discussion of the experimental evidence as to the strength of signals transmitted to great distances.

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